Math 246B Lecture 14 Notes

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1 Weierstrass's Theorem

1.1 Constructing holomorphic functions with a given zero set

Here is Weierstrass's theorem, which allows us to construct holomorphic functions with a prescribed zero set.

Theorem 1.1 (Weierstrass)**.** *Let $\Omega \subseteq \mathbb{C}$ be open, and let $A \subseteq \Omega$ be a set with no limit point in Ω. Assume that for any $a \in A$, we are given a positive integer $n(a)$. There exists $f \in \mathrm{Hol}(\Omega)$ such that $f^{-1}(\{0\}) = A$, and the multiplicity of each $a \in A$ is $n(a)$.*

Proof. We may assume that A is infinite and write $A = \{a_k, k = 1, 2, \dots\}$ with $a_k \neq a_{k'}$ if $k \neq k'$. Call $n_k := n(a_k)$. We shall try to construct f as an infinite product of the form

$$\prod_{k=1}^{\infty} (z - a_k)^{n_k} e^{g_k(z)},$$

where $g_j \in \mathrm{Hol}(\Omega)$ are chosen to achieve convergence.

Introduce the compact exhaustion $K_j = \{z \in \mathbb{C} : |z| \leq j, \mathrm{dist}(z, \mathbb{C} \setminus \Omega) \geq 1/j\}$. For each k, we have $a_k \in K_j$ for all j large enough. Define the sequence

$$j(k) = \begin{cases} 1 & a_k \in K_1 \\ \max\{j : a_k \notin K_j\} & a_k \notin K_1. \end{cases}$$

We have $j(k) \to \infty$ as $k \to \infty$: If $j(k) < M$ for some M, for infinitely many k, $a_k \notin K_{j(k)}$ for all large k. Then $a_k \in K_{j(k)+1} \subseteq K_M$ for infinitely many K, which cannot occur since A has no limit points in Ω.

We claim that for any k large enough, there exists $f \in \mathrm{Hol}(\Omega)$ such that $f_k^{-1}(\{0\}) = \{a_k\}$, the multiplicity of a_k is n_k, and such that there is a holomorphic branch g_k of $\log(f_k)$ in a neighborhood of $K_{j(k)}$. We have $a_k \notin K_{j(k)}$, so $|a_k| j(k)$ or $\mathrm{dist}(a_k, \mathbb{C} \setminus \Omega) < 1/j(k)$. We deal with each case:

1. $|a_k| > j(k)$: Take $f_k(z) = (z - a_k)^{n_k}$ and then take a holomorphic branch L_k of $\log(z - a_k)$ in $\mathbb{C} \setminus \{ta_k, t \geq 1\} \supseteq K_{j(k)}$. Then $g_k = n_k L_k$.

2. $\operatorname{dist}(a_k, \mathbb{C} \setminus \Omega) < 1/j(k)$: This distance is $\int_{z \in \mathbb{C} \setminus \Omega} |a_k - z|$, and pick $b_k \in \mathbb{C} \setminus \Omega$ such that $\operatorname{dist}(a_k, \mathbb{C} \setminus \Omega) = |a_k - b_k|$. This is the infimum of a continuous function over a closed set, and it goes to ∞ as $|z| \to \infty$, so the value is achieved; moreover, $b_k \in \partial\Omega$. Take
$$f_k(z) = \left(\frac{z - a_k}{z - b_k}\right)^{n_k} \in \operatorname{Hol}(\Omega).$$
Then $\{ta_k + (1-t)b_k : 0 \leq t \leq 1\} \cap K_{j(k)} = \varnothing$ because $\operatorname{dist}(ta_k + (1-t)b_k, \mathbb{C} \setminus \Omega) \leq t|a_k - b_k| < 1/j(k)$. Now the Möbius transformation
$$T(z) = \frac{z - a_k}{z - b_k}$$
maps $\mathbb{C} \setminus [a_k, b_k]$ to $\mathbb{C} \setminus \overline{R}_-$, and thus we can take $g_k(z) = n_k \operatorname{Log}(T_k(z))$, where this is the principal branch of T_k. So g_k is holomorphic in a neighborhood of $K_{j(k)}$.

This proves the claim.

Now any bounded component of $\mathbb{C} \setminus K_{j(k)}$ meets $\mathbb{C} \setminus \Omega$, so by Runge's theorrem, for any k, there is a holomorphic function $h_k \in \operatorname{Hol}(\Omega)$ such that $|g_k - h_k| \leq 2^{-k}$ on $K_{j(k)}$. Define $\tilde{f}_k := e^{-h_k} f_k \in \operatorname{Hol}(\Omega)$. Then $\tilde{f}_k$ does not vanish on $K_{j(k)}$. On $K_{j(k)}$, $\tilde{f}_k = e^{g_k - h_k}$, so (using $|e^z - 1| \leq |z| e^{|z|}$) we get $|\tilde{f}_k - 1| \leq 2^{-k} e$ on $K_{j(k)}$. If $K \subseteq \Omega$, then $K \subseteq K_{j(k)}$ for large k (as $j(k) \to \infty$ when $k \to \infty$), and this estimate shows that the infinite product
$$f = \prod_{k=1}^{\infty} f_k$$
converges locally uniformly and defines $f \in \operatorname{Hol}(\Omega)$ which solves the problem. □

1.2 Characterization of meromorphic functions

Weierstrass's theorem gives us an immediate way to characterize meromorphic functions.

Corollary 1.1. *Let g be meromorphic in Ω. Then $g = f/h$, where $f, h \in \operatorname{Hol}(\Omega)$.*

Proof. Let $h \in \operatorname{Hol}(\Omega)$ be such that the set of zeros of h agrees with the set of poles of g, with multiplicities. Then $f := gh \in \operatorname{Hol}(\Omega)$. □